A Low Statistics Estimation of the "212" keV/u ^{21}Na(p, γ)^{22}Mg Resonant Reaction Yield

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Abstract

A statistical estimate for the mean fusion yield, μ^*, for the "212" keV/u ^{21}Na(p, γ)^{22}Mg narrow resonance reaction has been determined by the method of Maximum Likelihood. The result obtained is $\mu^* = (\epsilon\rho)^{-1}(1.268 \pm 0.124) \times 10^{-12}$ ^{22}Mg/^{21}Na, with ϵ the γ-array efficiency and ρ the selected recoil charge state fraction. In this report I present my statistical analysis and discuss the statistical confidence of my result.

1 Introduction

Owing to the very small proton capture cross section of the "212" keV/u ^{21}Na(p, γ)^{22}Mg resonance reaction, the traditional assumption of Gaussian-like error distributions for the number of observed fusion recoils in *each* experimental run is not valid. Typically, for each run ($\sim$ 9 hours duration) < 9 coincidence recoil events are observed. It is *not* valid to assign statistical uncertainties of $\sigma = \pm\sqrt{N}$, with N the total number of coincident recoils to these data for the following reason:

- The "true" number of fusion events are purely Poisson in nature. The *estimated* rms for a Poisson distribution is the well known $\sigma = \sqrt{N}$. For small counts, this is only a crude estimate, as the Poisson distribution is highly skewed for small mean and for small total counts. In the limit that $N \to$ large, the Poisson distribution becomes symmetric and becomes essentially a Gaussian. Only in this limit, $\sigma = \pm\sqrt{N}$, is a valid estimator of the variance on the number of observed counts, owing to the symmetry of the Poisson distribution and the Central Limit Theorem[1].

I, therefore, cannot assume Gaussian-like error bars on the individual yield data points from *each* run and fit these points with the standard "least squares" method. What I can do instead is use the assumed Poisson statistics of each yield data point and the Maximum Likelihood Method[1, 2, 3, 4] (MLM) as the basis for my statistical analysis. Before proceeding the the MLM discussion, I first present the assumed statistical models for the various aspects of DRAGON: Poisson for the reactions, Binomial for γ detection, and Binomial for charge state selection.

2 The Statistics of the Data Set

It is my intent in this section to "walk" the reader through the logical steps leading up to the final statistical analysis. It is hoped from this pedantic approach that understanding, rather than confusion, will prevail.

As mentioned in point 1 in the previous section, the true number of fusion recoil events is Poisson distributed with respect to beam on target (this contrasts with the more familiar example of, say, radioactive decays being Poisson distributed with respect to time). More precisely, if μ is the mean true fusion reaction rate, in units of "recoils/10^{12} incident beam particles", and n is the true number of fusion events that occur during the course of a run, then the probability of those n events occurring, $P(n|\mu)$, is given by the Poisson distribution as,

$$P(n|\mu) = \frac{(fu)^n}{n!} \exp(-f\mu) \,, \tag{1}$$

where f is a constant to ensure the mean is normalized appropriately to the number of incident beam particles on target; that is, if we acquire $f \times 10^{12}$ beam particles on target, then our expected mean yield for this run will just be $f\mu$. The number f is obviously different for each experimental run as the total beam on target is different for each experimental run.

Ultimately, the goal of this analysis to obtain the best estimate of μ. We also, of course, can never directly observe the value of n; the true number of recoils, owing to:

- the γ-array efficiency $< 100\%$
- charge state distribution selection

(if we could directly observe n, μ would be immediately determined, given beam on target, and our lives would be so much simpler).

Prompt γ emission follows the production of a fusion recoil. When n true fusion events take place, the γ-array is presented with n emission gammas. Suppose the γ-array has an overall detection efficiency of ϵ. We then ask: *"What is the probability $P(g|n,\epsilon)$ that, of n gammas presented, the γ-array detects g ($g \leq n$) of them?"* The answer to this question is provided by the Binomial probability distribution [2],

$$P(g|n,\epsilon) = \frac{n!}{g!(n-g)!}\epsilon^g(1-\epsilon)^{n-g} \,. \tag{2}$$

Of g gammas detected by the γ-array, g fusion recoils make their way to the downstream exit of the target to MD1. Only those recoils of the selected charge state will be "accepted" by the recoil separator; the rest being rejected by MD1. If we know the probability, ρ, for our selected charge state, then we can ask the following: *"Of g recoils presented to MD1, what is the probability, $P(r|g,\rho)$, that $r(r \leq g)$ recoils are accepted by the separator for eventual detection at the end of DRAGON?"* Again, the answer is the Binomial distribution,

$$P(r|g,\rho) = \frac{g!}{r!(g-r)!}\rho^r(1-\rho)^{g-r} \ . \tag{3}$$

The expectation value of r, r^*, is what we measure in our coincidence DSSSD data. We, of course, have no way of directly observing g in our experiment.

Having laid out the statistical distributions from which the experiment samples from, I describe in the next section how this is all put to use in arriving at the best estimate, μ^*, for μ.

3 How to Proceed?

As mentioned in § 2, we have no means to directly measure n, or g, for a given experimental run. We can, however, find their estimates. The expectation value of the Binomial distribution in equation 3 is what we measure in the DSSSD. That is,

$$r^* = \sum_{r=0}^{g} rP(r|g,\rho) = \rho g \ . \tag{4}$$

Rearranging the above result to determine g, we can determine the best estimate for the unknown value of n in equation 2. The result is,

$$g^* = \sum_{g=0}^{n} gP(g|n,\epsilon) = \epsilon n \tag{5}$$

Setting $g^* = g$ and grouping terms in equations 4 and 5 produces,

$$n \equiv n^* = \frac{r^*}{\rho\epsilon} \ , \tag{6}$$

which certainly appeals to our intuitive expectations.

With this result, I now move on to the details of the determination of best estimate of the mean reaction yield, μ, in the next section.

4 Maximum Likelihood:What it It?

Consider the following: We have a physical quantity we want to measure; μ in the case of this experiment. Prior to making any measurement of μ, our knowledge of what the value is μ is one of complete ignorance. [1] A distribution that would represent this state of knowledge (or lack thereof) would be something like that in frame a) of figure 1; all we can say about μ is that it is some real number greater than zero.

We make our first measurement of μ_1 and form a more refined distribution function for it. After one measurement on μ our state of knowledge has improved from complete ignorance to something like that shown in the yellow distribution of frame b) of figure 1; a broad distribution whose maximum, μ^*, that we now consider our current best estimate of the true value of μ.

Now we make a second measurement of μ_2 and we obtain another distribution, similar to that for μ_1, but slightly different in that $\mu_1 \neq \mu_2$ in general. The distribution for μ_2 is shown in frame b) of figure 1 in colour cyan.

The joint probability distribution of these two measurements will be the product of their two distributions. We can imagine that if we take the product of the yellow and cyan distributions in frame b) of figure 1 we will obtain a distribution like that shown in frame c) of figure 1. As we acquire more measurements of μ, forming the set $S = \{\mu_1, \mu_2, \ldots, \mu_N\}$, we continue to combine the individual probability distributions for each μ_i by taking their

[1] For the purposes of this discussion, I am intentionally neglecting "knowledge" proffered up by theoretical estimates.

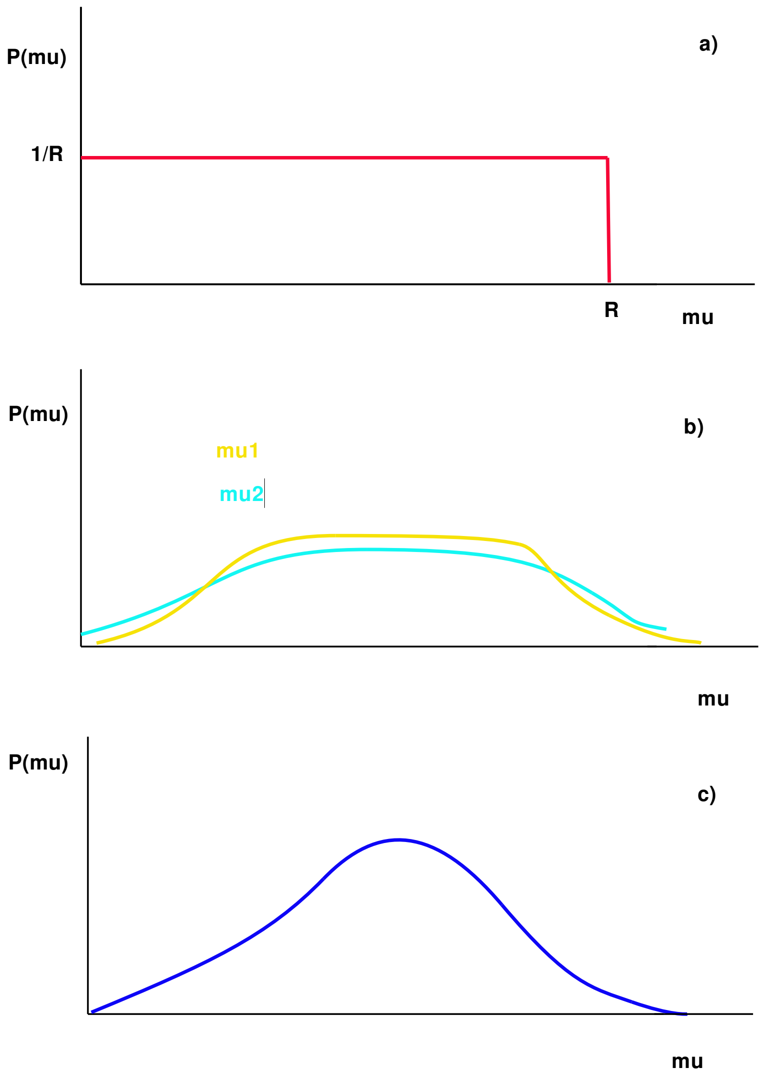

Figure 1: Sequence of improving knowledge: Frame a) shows a representation of complete ignorance on the value of μ; all values of μ are equally probable. Frame b) shows two separate measurements of μ that are, individually, broad. Frame c) shows what the joint probability distribution looks like when combining the distributions of μ_1 and μ_2.

products. This will lead to a joint probability distribution function that will be sharply peaked and it is the value of μ at the maximum of this distribution that we take as our best estimator of the value of μ.

More formally, if $P_i(n_i|\mu)$ is the probability distribution of the i'th measurement of n, then the joint probability distribution $\mathcal{L}$ of μ will be:

$$\mathcal{L}(\mu) = \prod_{i=1}^{N} P_i(n_i|\mu) \; , \tag{7}$$

with proviso,

$$\sum_{n_i=0}^{\infty} P_i(n_i|\mu) = 1$$

that is, each probability distribution is normalized with respect to the *measured* parameter; a condition clearly satisfied with P_i a Poisson distribution.

We want to maximize equation 7. Finding the maximum of this product is messy. An equivalent problem to maximizing equation 7 is to instead maximize: [2]

$$\ln(\mathcal{L}) = \sum_{i=1}^{N} \ln(P_i(n_i|\mu)) \; . \tag{8}$$

Substituting the expression for $P_i(n_i|\mu)$ from equation 1 into equation 8 and taking the partial derivative with respect to μ results in,

$$\frac{\partial}{\partial\mu} \sum_i [n_i^* \ln(f_i\mu) - f_i\mu - \ln(n_i!)] = 0 \tag{9}$$

which yields the analytic solution,

$$\mu^* = \frac{\sum_i n_i^*}{\sum_i f_i} = (\epsilon\rho)^{-1} \frac{\sum_i r_i^*}{\sum_i f_i} \; , \tag{10}$$

with the result of equation 6 used in the last step.

[2]Recall, $\frac{d}{dx}[\ln f(ax)] = a\frac{f\prime(ax)}{f(ax)} = 0$ implies the derivative is equal to zero – which is just the condition for maximization.

A few useful points about the asymptotic (large N) behaviour of likelihood estimators:

1. They are unbiased.

2. They are minimum variance.

3. They tend to normal distributions; that is,

$$\mathcal{L} \sim \exp\left\{-\frac{1}{2}\frac{(\mu-\mu^*)^2}{\sigma^2}\right\}.$$

Because of point 3, the 1σ variance of the likelihood parameter can be estimated by either:

- Finding the values of μ where $\ln\mathcal{L}$ changes by -1/2 from its maximum value, $\ln\mathcal{L}_{\max} - 1/2$, or

- Taking the second partial derivative evaluated at μ^*:

$$\sigma = \left(\frac{\partial\ln(\mathcal{L})}{\partial\mu}\right)^{-1/2}\Bigg|_{\mu^*}.$$

With this introduction, let us now finally get to the results.

5 Results

A total of 18 runs were analyzed from the "212" keV/u resonance study, the results from which are displayed in table 1 in Appendix A. One thing I would like to point out and explain is the apparent pattern of percentage uncertainties in the beam on target values as a function of beam energy. The reason for this "pattern" is due to the difference in the uncertainties in the proportionality factors, R, used to scale total elastics counts into total beam on target. The relative uncertainties for these scaling factors are shown in table 2; note the close agreement between their respective ratios and the ratios of percentage uncertainties of beam on target between runs at different beam energies.

The ln-likelihood and likelihood functions for these 18 runs are shown in figure 2. The top frame of the figure shows an expanded view of a plot of the natural logarithm of the likelihood function, $\ln(\mathcal{L})$ as a function of μ; the lower frame a plot of the likelihood function, $\mathcal{L}$ as a function of μ. The maximum of $\mathcal{L}$ occurs at $\mu^* = 1.268$, with 1σ confidence interval of $\sigma = \pm 0.122$. As is clearly evident from the plots in figure 2, the likelihood function for this data set has essentially achieved asymptotic normality; the top plot of of the ln-likelihood being parabolic; the lower plot of the likelihood being Gaussian in shape.

Note that equation 10 admits the possibility of directly including the statistical errors on the values of f_i; the total beam in run i as a fraction of 1×10^{12}. It is found that the *relative* error on $\sum_i f_i$ is 0.015. When added in quadrature with the 1σ value on μ^* of ± 0.122, the total statistical uncertainty on μ^* is found to be ± 0.124.

Figure 3 shows a yield plot of the data runs displayed in table 1. The vertical bars attached to each yield point indicates the 1σ confidence interval for that data point, obtained by solving the integral equation,

$$\sigma_i = \int_{\mu_i^* - a}^{\mu_i^* + b} P_i(n_i|\mu_i)\, d\mu_i \ , \tag{11}$$

for a, b subject to the proviso,

$$P_i(n_i|\mu_i^* - a) = P_i(n_i|\mu_i^* + b) \ ,$$

and $\mu_i^* = Y_i$; Y_i being the yield of the i'th run.

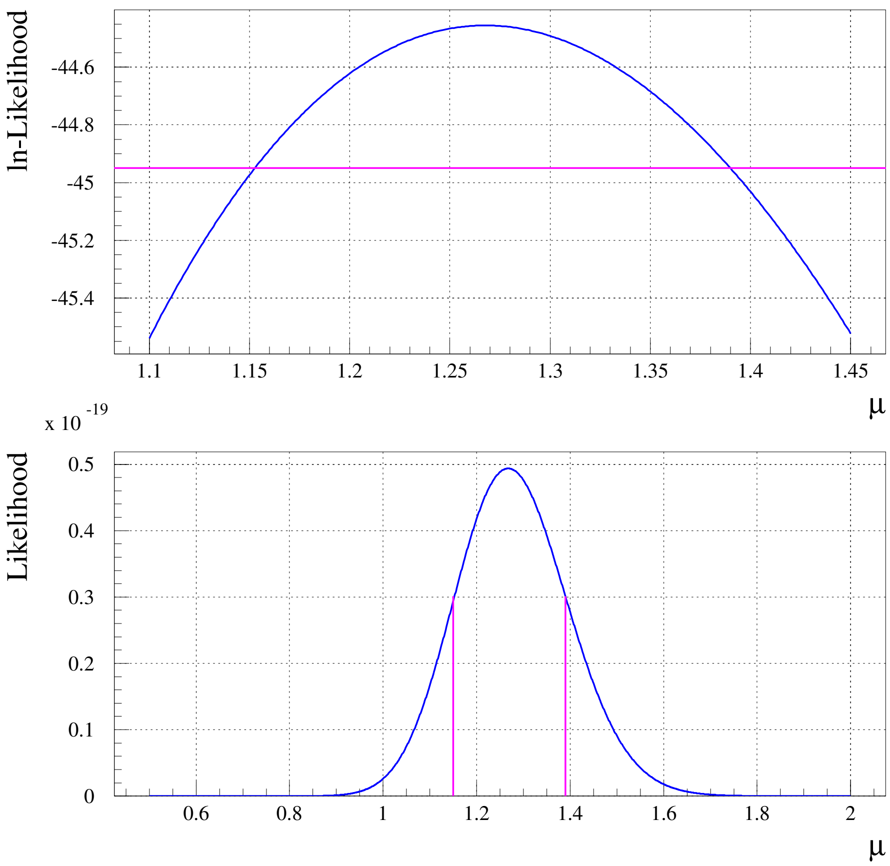

Figure 2: Plot of the Likelihood function. The top frame shows an expanded view of the ln-likelihood as a function of μ. The lower frame shows a global view of the standard likelihood as a function of μ. The maximum occurs at $\mu^* = 1.268$. The horizontal purple line denotes where ln-likelihood changes by 0.5; the vertical purple lines denote the 1σ confidence interval on μ^* of $\sigma = \pm 0.122$.

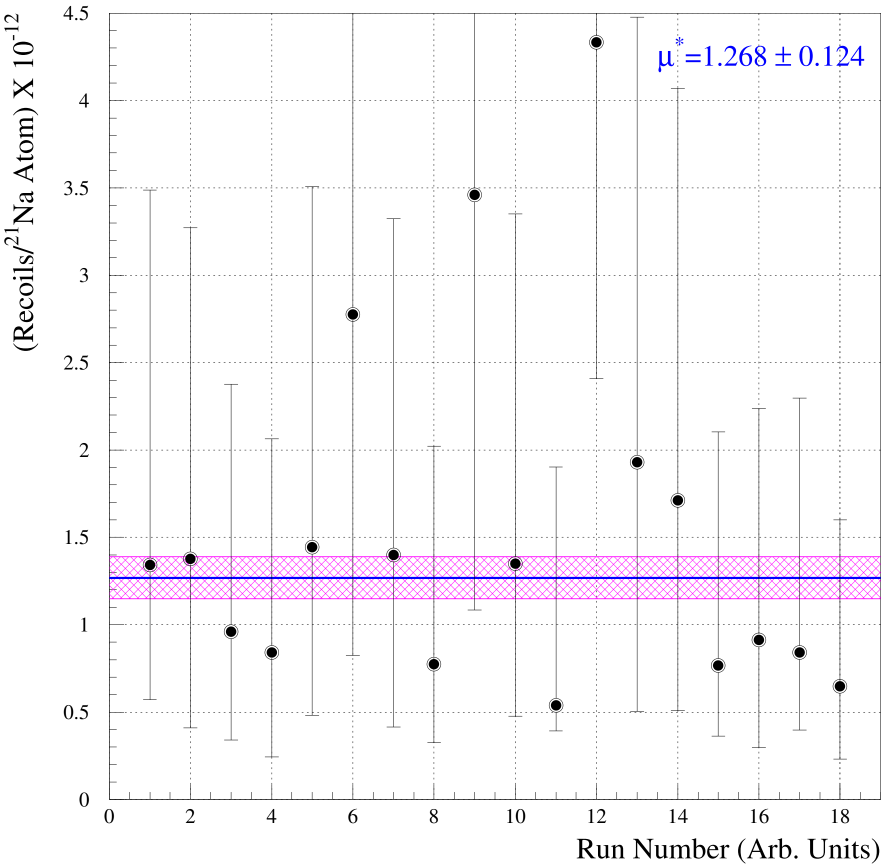

Figure 3: Plot of the yield points for the 18 data runs shown in table 1. The 1σ confidence intervals for each data point is shown by the vertical bars attached to each point. The blue line is the maximum likelihood estimate of $\mu^* = 1.268$. The hatched region denotes the combined maximum likelihood 1σ confidence interval and statistical uncertainty from $\sum_i f_i$ on μ^*, as described in § 5.

6 Conclusion

I have presented arguments for why application of the "standard" rule of $\sigma = \pm\sqrt{N}$ is not valid for low statistics events. The observation of 1 count is clearly *not* consistent with zero, as $\pm\sqrt{N}$ would incorrectly imply; I liken this to an analogy of chocolate chips being found to be distributed in cookies according to a Poisson distribution. Your observation of 1 chocolate chip in a cookie is, in and of itself, clearly inconsistent with a null result.

The data has been analysed using the MLM and this analysis has produced a mean yield of $Y = (\epsilon\rho)^{-1}(1.268 \pm 0.124) \times 10^{-12}\ {}^{22}\mathrm{Mg}/{}^{21}\mathrm{Na}$, with ϵ the as yet to be determined γ-array efficiency, and ρ the selected charge state fraction of the ^{22}Mg recoils.

The MLM is quite general, and can easily accomodate the possibility of a position dependent γ-array efficiency on z along the beam direction; simply subscript ϵ with respect to run number throughout this work.

Finally, it is evident from figure 3 that there are no systematic pathologies in our data set; the MLM esitmate of μ is qualitatively reasonable with the bare arithmetic average of the yeilds of 1.535, and the 1σ confidence intervals of all but one data point are consistent with the MLM estimate of μ.

A Appendix

Run #	Energy (keV/u)	Recoils	Beam ($\times 10^{12}$)	σ_{beam}/Beam %
4878	215	4	2.978	7.05
4879	215	9	6.522	7.02
4880	215	6	6.224	7.02
4873	220	8	9.154	4.08
4883	220	7	4.850	4.09
4884	220	9	3.235	4.23
4885	220	9	6.418	4.09
4886	220	4	5.122	4.09
4887	220	8	2.308	4.14
4893	220	6	4.410	4.20
4904	220	1	1.828	4.39
4905	220	2	0.4515	5.05
4906	220	12	6.206	4.11
4907	220	9	5.240	4.13
4867	225	3	3.844	7.01
4868	225	7	7.566	6.89
4869	225	3	3.524	6.94
4872	225	6	9.262	6.88

Table 1: Summary of the 18 data runs used in the ^{21}Na(p, γ)^{22}Mg "212"keV/u resonance analysis. All quoted recoil numbers are for those recoils with coincidence γ energies greater than or equal to 3.0 MeV.

σ_R^{215}/R^{225}	σ_R^{220}/R^{225}	σ_R^{225}/R^{225}
7.0×10^{-2}	4.06×10^{-2}	6.85×10^{-2}

Table 2: Listing of the relative uncertainties of the scaling factors used to scale total elastic monitor counts into total beam on target values. Note each scaling factor was applied to each run within its respective beam energy value.

References

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