2.3. EXPRESSIONS FOR γ-RAY ANGULAR DISTRIBUTIONS

In the following subsections, general expressions for the γ-ray angular distributions of direct capture (primary) and subsequent nuclear decay (secondary) transitions will be discussed. Illustrative examples, relevant to the subsequent experiments, will also be presented. The derivation of these expressions is described in the appendices.

2.3.1. Direct capture transitions. The angular distribution for the direct capture γ-ray transition is given (appendix A.1.1) by:

Primary

$$W(\vartheta) = \sum (l_i 0 l_i 0 | k 0) Z_1(L l_i L l_i; l_f k) Q_k P_k(\vartheta). \tag{11}$$

These expressions depend only on the orbital angular momenta l_i and l_f of the initial and final state, respectively and on the multipole order, L, of the γ-ray transition. They are independent of the total spin J of the final state as well as of the intrinsic spins of the target nucleus j_t and projectile j_p (i.e. the channel spin $S = j_p + j_t$). Examples of the most common angular distributions observed in the present work are:

$$\mathrm{E1}(p \to s) \qquad W(\vartheta) = 1 - P_2(\vartheta) \propto \sin^2\vartheta,$$
$$\mathrm{E1}(p \to d) \qquad W(\vartheta) = 1 - \tfrac{1}{10} P_2(\vartheta) \propto 1 + \tfrac{1}{6}\sin^2\vartheta,$$
$$\mathrm{E1}(f \to d) \qquad W(\vartheta) = 1 - \tfrac{2}{5} P_2(\vartheta) \propto 1 + \sin^2\vartheta.$$

A sensitive test of the above theoretical angular distributions is provided by the study of the direct capture process in the two reactions $^{16}O(p,\gamma)^{17}F$ and $^{17}O(p,\gamma)^{18}F$. The angular distributions to final states with the same orbit l_f should be identical in the two reactions despite the different target spins ($J_t = 0$ and $\frac{5}{2}$) as well as the different possible total spins ($J_f = S + l_f$). Examples are described in subsects. 4.1 and 4.2.3.

The summation over the magnetic indices can be performed with the help of the usual relations between CG and Racah coefficients and results in

Secondary

$$W(\vartheta_2) = \sum (l_1 0 l_1 0 | k 0) W(l_1 l_2 l_1 l_2; L_1 k) \times W(J_2 l_2 J_2 l_2; Sk) Z_1(L_2 J_2 L_2 J_2; J_3 k) P_k(\vartheta_2). \tag{A.32}$$

primary unobserved

In the case of a mixture of (L_2, L_2^*) multipoles in the secondary transition, it is straightforward to generalize the above expression to that given in subsect. 2.3.2 (eq. (14)). If the direct-capture transition (unobserved primary) can proceed from several partial waves l_1 to several orbits l_2 in the intermediate state, then the final angular distribution for the secondary transition is given by an incoherent sum over the individual components:

$$W(\vartheta_2) = \sum_{l_1 l_2} \sigma_{l_1 l_2} W_{l_1 l_2}(\vartheta_2). \tag{A.33}$$

Coefficients and functions used in angular correlation analysis

1. Coefficients

1.1. GENERAL

The basic coefficients required for angular correlation analysis are the Clebsch–Gordan, Racah and 9–j coefficients. These are not tabulated in the present work, but fairly extensive tabulations are available (see § 1.7 below). The widespread appearance of electronic digital computers with sophisticated program compilers makes it fairly easy to compute coefficients as they are required and removes in some degree the need for extensive tabulations. An advantage of this situation is that coefficients beyond the range of existing tables are readily obtainable. The many parameters of a 9–j symbol render impossible any attempt at a complete tabulation even if reasonable limitations on the spins are imposed. Computer programs that analyse measured correlations by, for example, the least squares method, can readily contain subroutines to calculate the required coefficients so that the user need only specify the sets of spins to be tested.

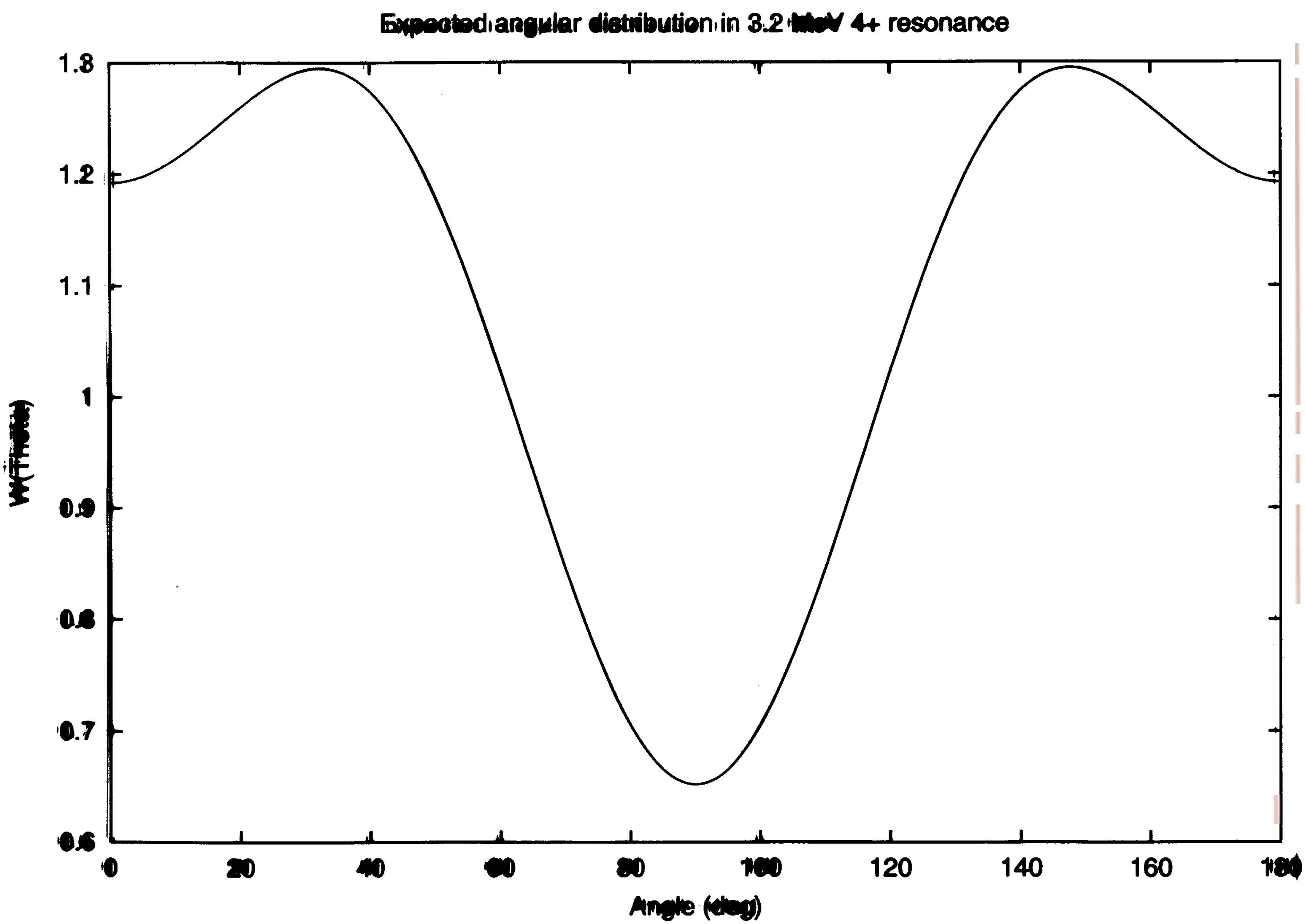
The 3–j and 6–j symbols are coefficients equivalent to the Clebsch–Gordan and Racah coefficients respectively but are more symmetrical than the latter. There has been a trend toward the use of these coefficients in the literature. The unsymmetric coefficients have been used in the present work, and their relation to the symmetric ones is indicated below.

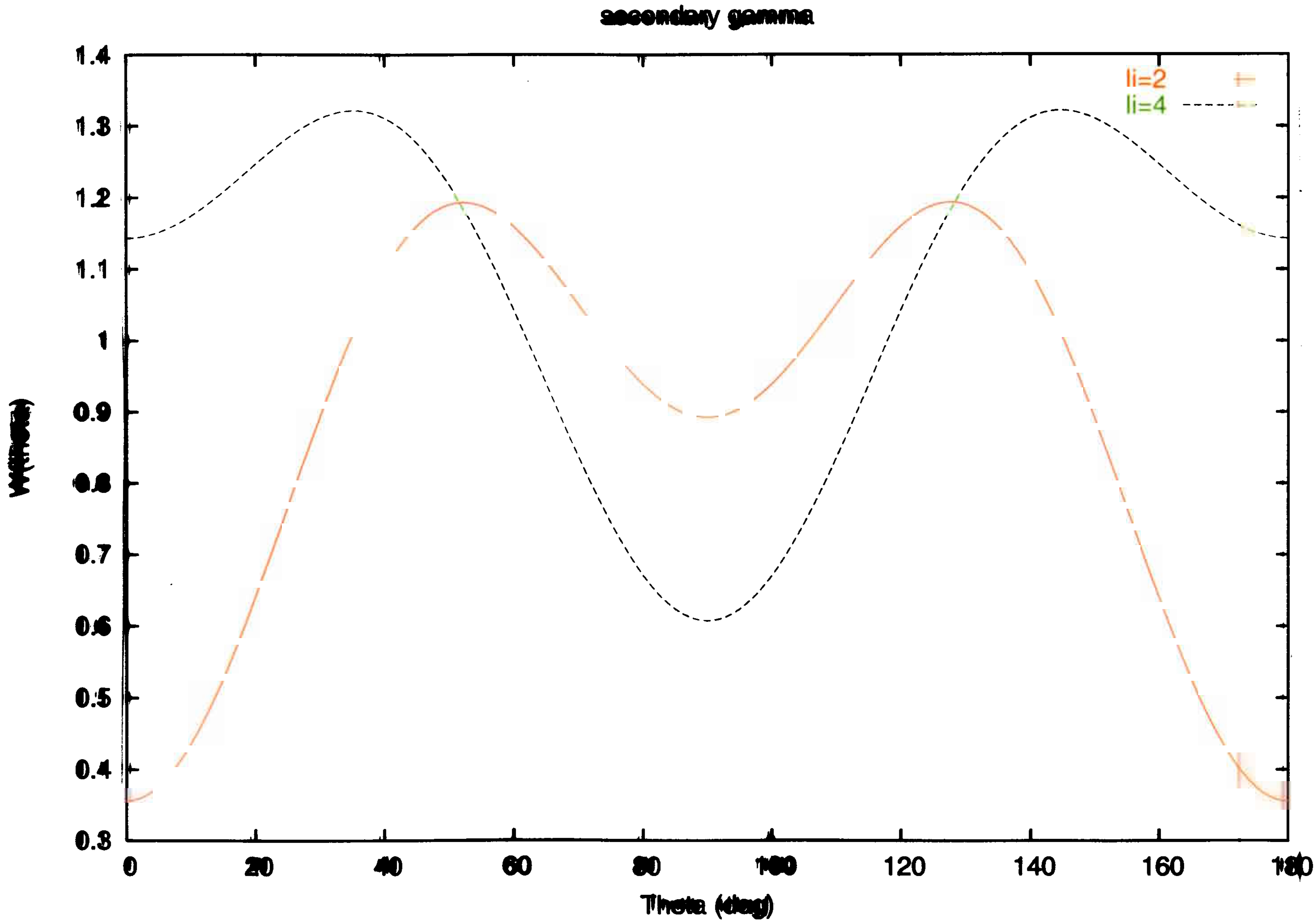
1.2. Z AND Z_1 COEFFICIENTS

The Z and Z_1 coefficients are defined by eqs. (3.3) and (3.4) which are repeated below

$$Z(lbl'b';ak) = \hat{l}\hat{l}'\hat{b}\hat{b}'(l0,l'0|k0)\,W(lbl'b';ak)$$

$$Z_1(LbL'b';ck) = (-)^{k-L+L'-1}\,\hat{L}\hat{L}'\hat{b}\hat{b}'(L1,L'-1|k0)\,W(LbL'b';ck).$$


Expected angular distribution in 3.2 MeV 4+ resonance
W(Theta)
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0
20
40
60
80
100
120
140
160
180
Angle (deg)



secondary gamma
li=2
li=4
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0
20
40
60
80
100
120
140
160
180
Theta (deg)

The X coefficient arises in the relation between different ways of combining the vectors a, b, d, e to give a resultant k. In terms of Clebsch-Gordan coefficients, it is given by

$$(ab|c)(de|f)(cf|k) = \sum_{g,h} (2g+1)^{\frac{1}{2}}(2h+1)^{\frac{1}{2}}(2c+1)^{\frac{1}{2}}(2f+1)^{\frac{1}{2}}(ad|g)(be|h)(gh|k)\ X\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$

The X coefficient may also be expressed in terms of Racah coefficients:

$$X\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = \sum_{z} (2z+1)W(abkf,cz)W(dfhb,ez)W(adkh,gz) .$$

From either definition, it follows that X is defined only if the parameters in each row and each column form the sides of a triangle with integral sum. The X coefficient is unchanged by interchanging the roles of rows and columns, while interchange of any two rows or any two columns multiplies X by $(-1)^{a+b+c+d+e+f+g+h+k}$. As a consequence of the latter symmetry, if two rows or two columns are the same, X vanishes unless the third has even sum.

If any parameter vanishes, the X coefficient may be permuted into

$$X\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & 0 \end{pmatrix} = (-1)^{g+e-a-e}\ (2g+1)^{-\frac{1}{2}}(2e+1)^{-\frac{1}{2}}\ W(abde,eg)\ \delta_{cf}\delta_{gh}$$

and for (p,α,p) (transition specified by $s_1 \ell_1 j_1 \ell_{12} j_2 \ell_2 s_2$)

$$W_{tt'} = \sum_{k_1 k_2 k_{12}} (-1)^{\xi} Z(\ell_1 j_1 \ell_1' j_1', s_1 k_1) G_0 \begin{pmatrix} j_1 & \ell_{12} & j_2 \\ k_1 & k_{12} & k_2 \\ j_1' & \ell_{12}' & j_2' \end{pmatrix} Z(\ell_2 j_2 \ell_2' j_2', s_2 k_2) A_{k_1 k_2 k_{12}}$$

with $\xi = s_1 + s_2 + j_1 + j_2$.

To obtain the (γ,α,p) and (γ,α,γ) correlations only formal changes are needed.

(c) Intermediate Gamma-Radiation

We assume here that R_{12} is a gamma-ray.

$$\text{Define } G_1 \begin{pmatrix} j_1 & L_{12} & j_2 \\ k_1 & k_{12} & k_2 \\ j_1' & L_{12}' & j_2' \end{pmatrix} = R\left(i^{-L_{12}+\Pi_{12}-L_{12}'-\Pi_{12}'-k_1-k_2+2}\right)$$

$$\times (2L_{12}+1)^{\frac{1}{2}} (2L_{12}'+1)^{\frac{1}{2}} (2k_1+1)^{\frac{1}{2}} (2k_2+1)^{\frac{1}{2}} (L_{12} L_{12}' - 11 | k_{12} 0) X \begin{pmatrix} j_1 & L_{12} & j_2 \\ k_1 & k_{12} & k_2 \\ j_1' & L_{12}' & j_2' \end{pmatrix}$$

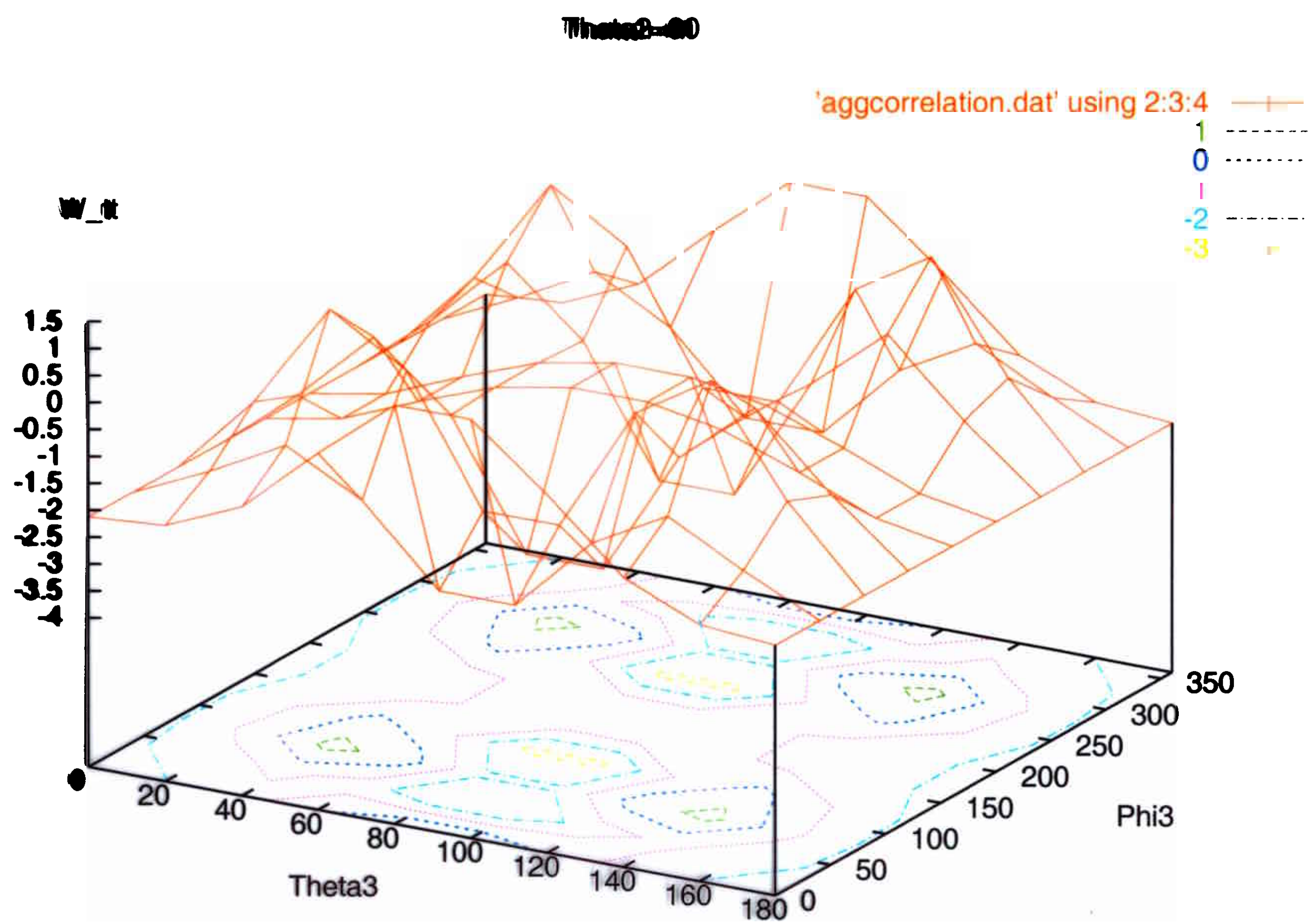
triple correlation

For (p,γ,γ) then (transition specified by $s_1 \ell_1 j_1 L_{12} j_2 L_2 I_2$)

$$W_{tt'} = 4 \sum_{k_1 k_2 k_{12}} (-1)^{\psi} Z(\ell_1 j_1 \ell_1' j_1', s_1 k_1) G_1 \begin{pmatrix} j_1 & L_{12} & j_2 \\ k_1 & k_{12} & k_2 \\ j_1' & L_{12}' & j_2' \end{pmatrix} Z_1(L_2 j_2 L_2' j_2', I_2 k_2) A_{k_1 k_2 k_{12}} (\theta \phi_1, \theta \phi_2, \theta \phi_3)$$

with $\psi = s_1 + I_2 + \ell_1 + \ell_1' + L_{12} + L_{12}' + L_2 + L_2' + j_1 + j_2$.

For (γ,γ,γ) we need only replace $s_1 + \ell_1 + \ell_1'$ by $I_1 + L_1 + L_1'$ and $Z(\ell_1 j_1 \ell_1' j_1', s_1 k_1)$ by $Z_1(L_1 j_1 L_1' j_1', I_1 k_1)$ and divide by 2.


'aggcorrelation.dat' using 2:3:4
1
0
-2
-3
W_tt
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
20
40
60
80
100
120
140
160
180
Theta3
0
50
100
150
200
250
300
350
Phi3

one finds that

$$W_{tt'} = \sum_{k_1k_2k_{12}} (-1)^{\lambda} g\, c_{k_10}(L_1L_1')^* c_{k_20}(L_2L_2')^* c_{k_{12}0}(L_{12}L_{12}')$$

$$\times\, W(L_1j_1L_1'j_1', I_1k_1)W(L_2j_2L_2'j_2', I_2k_2)\ X\begin{pmatrix} j_1 & j_2 & L_{12} \\ j_1' & j_2' & L_{12}' \\ k_1 & k_2 & k_{12} \end{pmatrix} \Lambda_{k_1k_2k_{12}}$$

where $\lambda = I_1+I_2+L_1+L_2+L_{12}+L_{12}'-j_1-j_2+k_{12}$

$$g = \left[(2j_1+1)(2j_1'+1)(2j_2+1)(2j_2'+1)(2k_1+1)(2k_2+1)\right]^{\frac{1}{2}}$$

$$c_{k0}(LL') = \sum_{M} (-1)^{L-M} a(LM)^* a(L'M)(LL'-MM|k\,0)$$

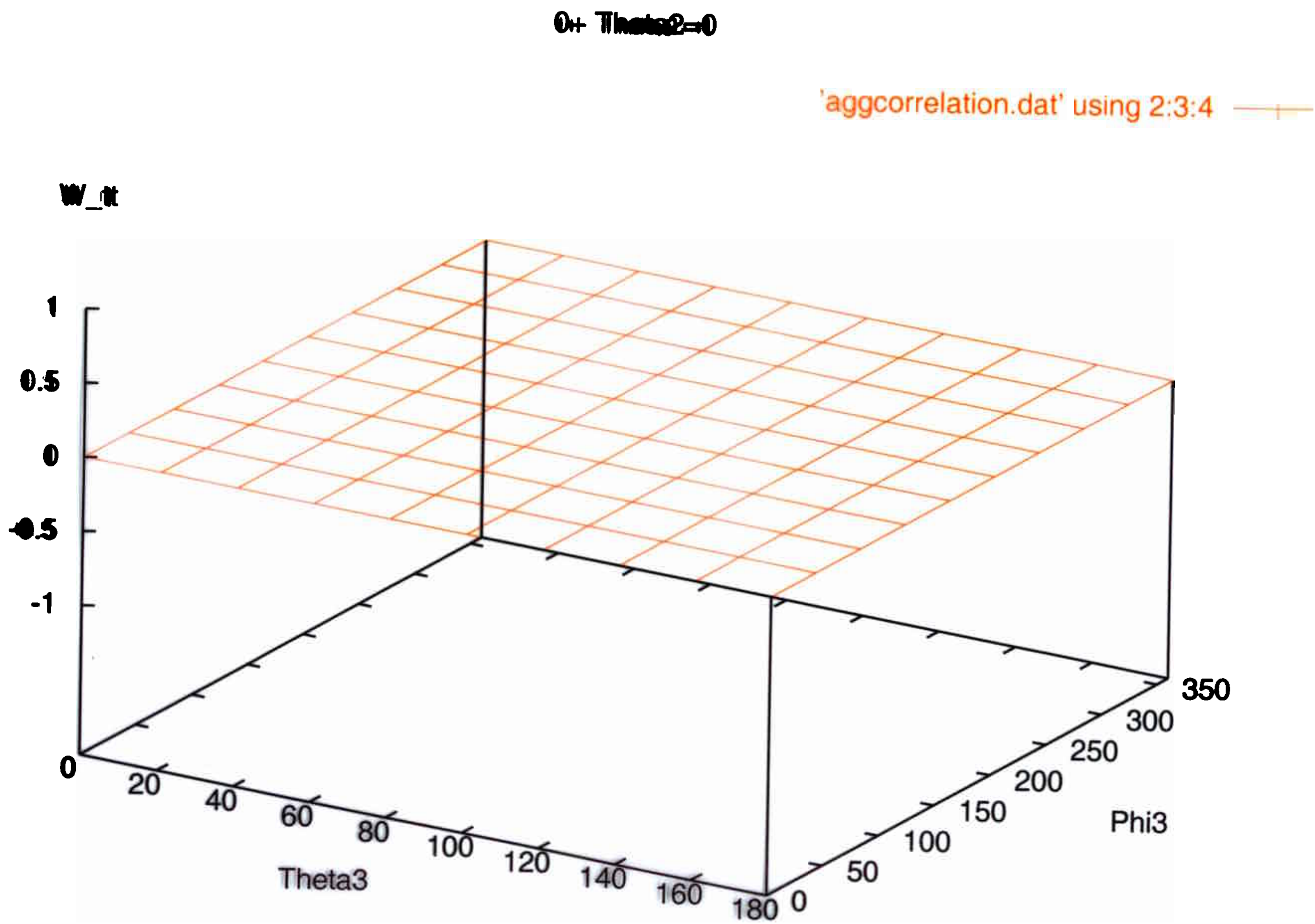
$$\Lambda_{k_1k_2k_{12}} = \sum_{\mu_1\mu_2} (k_1k_2\mu_1\mu_2|k_{12}\ \mu_1+\mu_2)\left[\frac{(4\pi)^3}{(2k_1+1)(2k_2+1)(2k_{12}+1)}\right]^{\frac{1}{2}}$$

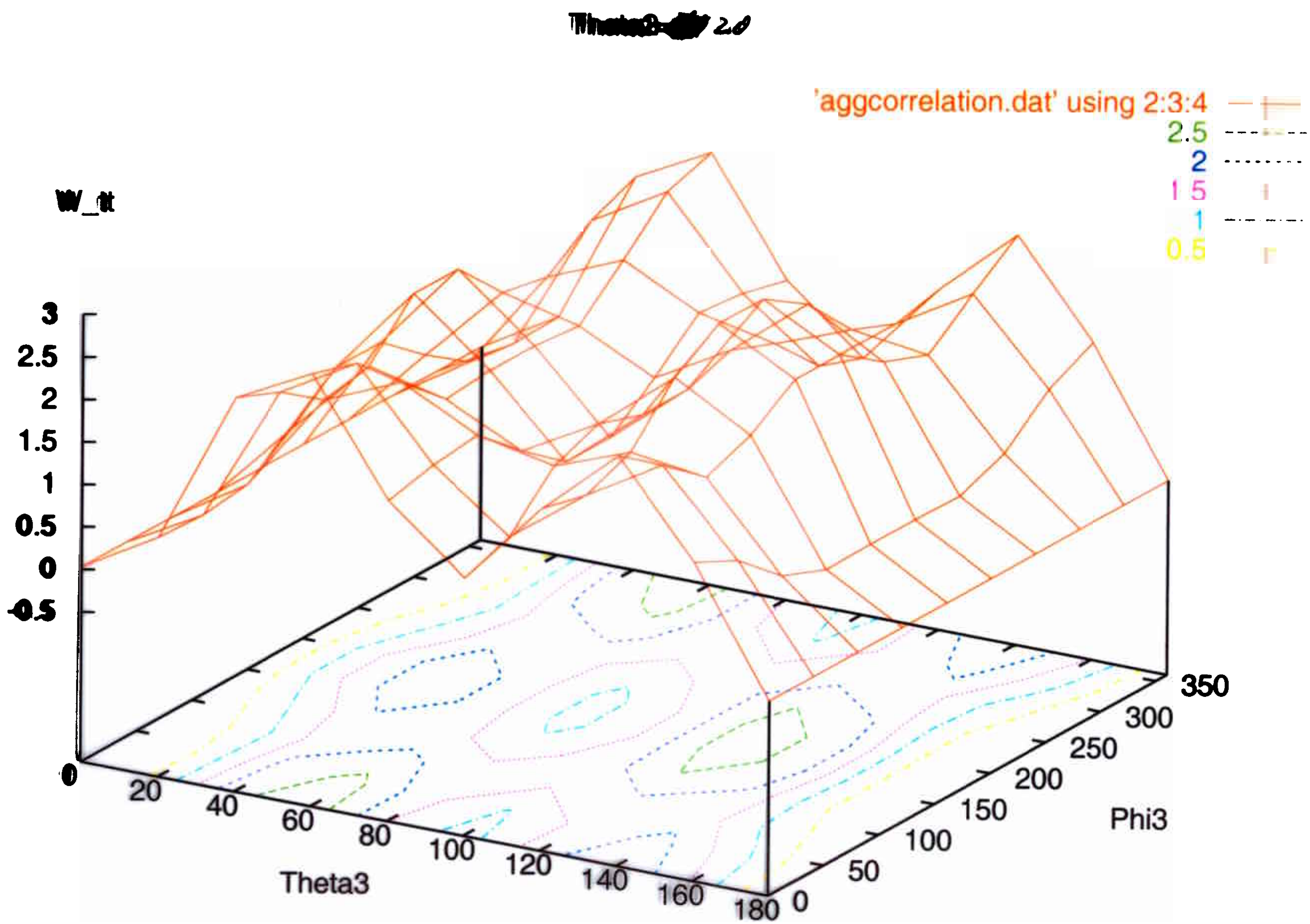
$$\times\, Y_{k_1}^{\mu_1}(\theta_1\phi_1)Y_{k_2}^{\mu_2}(\theta_2\phi_2)Y_{k_{12}}^{\mu_1+\mu_2\,*}(\theta_{12}\phi_{12})$$

The $Y_\ell^m(\theta,\phi)$ are the usual spherical harmonics, so normalized that

$$Y_\ell^m(\theta,\phi) = (-1)^{\frac{1}{2}(m+|m|)}\sqrt{\frac{(\ell-|m|)!}{(\ell+|m|)!}}\, e^{im\phi}\sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell^{|m|}(\cos\theta)$$

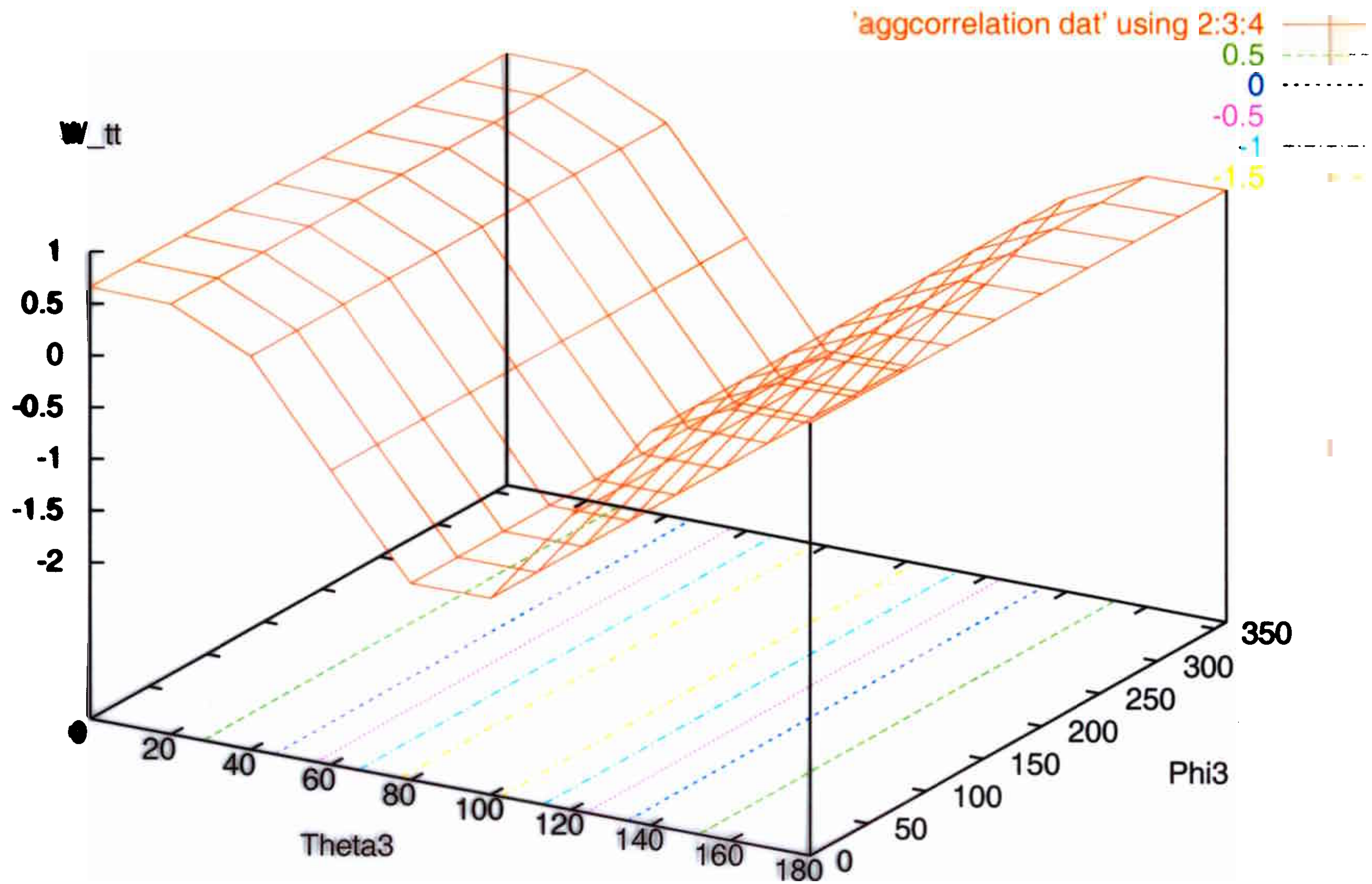
where the $P_\ell^{|m|}$ are the associated Legendre functions of the first kind. The summation is to be extended over all values of $k_1k_2k_{12}$ consistent with the triangular conditions for the W and X coefficients and the properties of the $a(LM)$.


0+ Theta2=0
'aggcorrelation.dat' using 2:3:4
W_tt
1
0.5
0
-0.5
-1
0
20
40
60
80
100
120
140
160
180
Theta3
0
50
100
150
200
250
300
350
Phi3



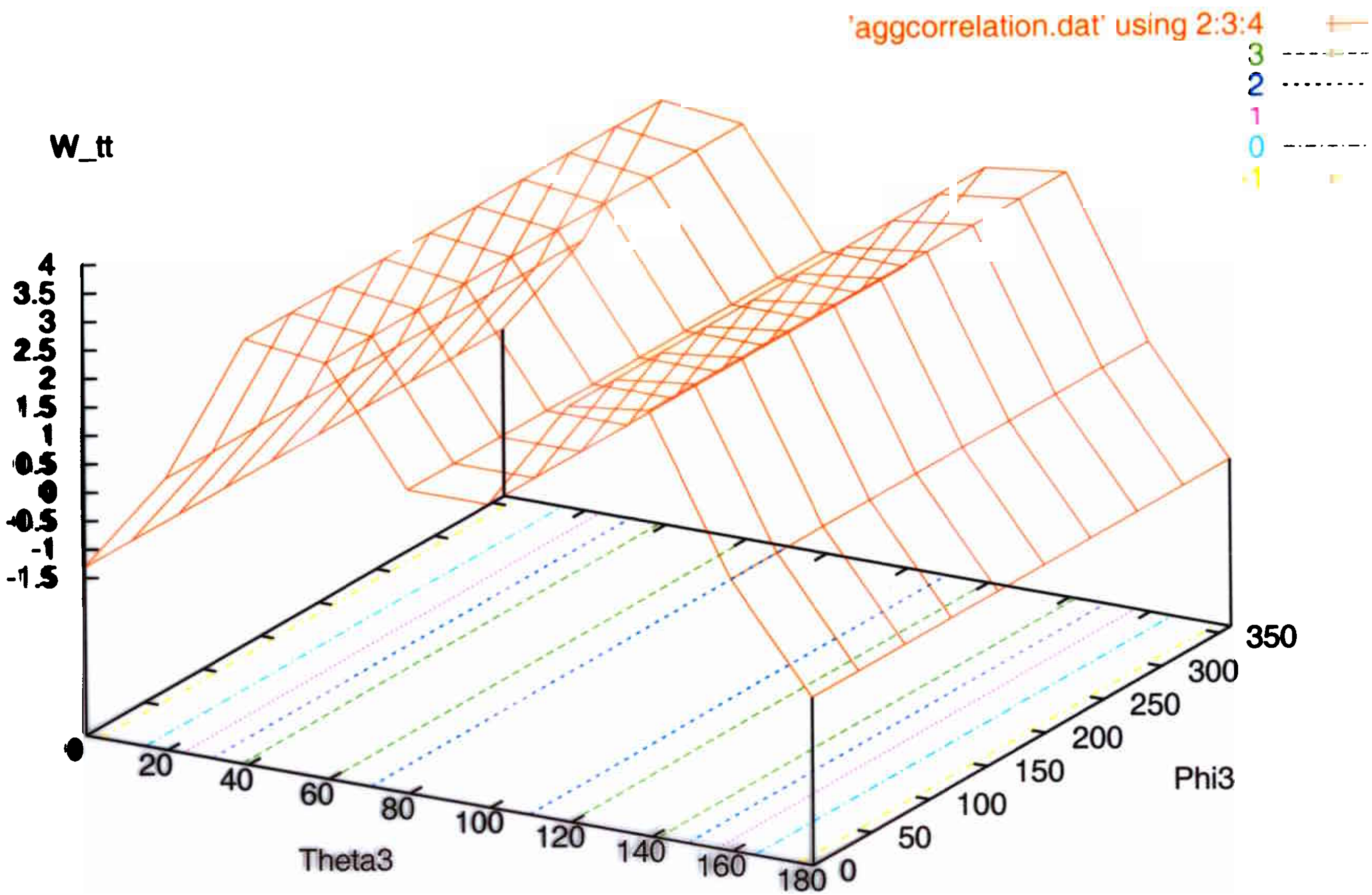
'aggcorrelation.dat' using 2:3:4
2.5
2
1.5
1
0.5
W_tt
3
2.5
2
1.5
1
0.5
0
-0.5
0
20
40
60
80
100
120
140
160
180
Theta3
0
50
100
150
200
250
300
350
Phi3

2+ Theta3=0



'aggcorrelation dat' using 2:3:4
0.5
0
-0.5
-1
-1.5
W_tt
1
0.5
0
-0.5
-1
-1.5
-2
0
20
40
60
80
100
120
140
160
180
Theta3
0
50
100
150
200
250
300
350
Phi3

Theta2=0



'aggcorrelation.dat' using 2:3:4
3
2
1
0
-1
W_tt
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
20
40
60
80
100
120
140
160
180
Theta3
0
50
100
150
200
250
300
350
Phi3